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# КУЛОН-ЯДЕРНАЯ ИНТЕРФЕРЕНЦИЯ (КЯИ) В УПРУГОМ РАССЕЯНИИ ПРОТОНОВ

(бакалаврская работа)

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# Contents

## Annotation

A new consistent derivation of the formula for the modulus of the proton-proton scattering amplitude is given, which is designed to retrieve information on the strong interaction phase due to the Coulomb-nuclear interference.

## Acknowledgement

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And above all, the Almighty God, the author of knowledge and wisdom, for his showers and blessings. Thank you.

# Chapter 1

# Introduction

The thesis concerns one special aspect of the problem of phase of the strong ("nuclear") interaction scattering amplitude, $T_N(s,t)$, with $\sqrt{s}$ the c.m.s collision energy and $t$ the square of the transferred momentum. Physical significance of the scattering phase for understanding the space-time picture of the high-energy scattering was discussed in Ref.[1].

The problem is that if only strong interaction would occur then the phase could not be detected because we would deal with the modulus $\mid T_N \mid$ only as the measured cross sections are proportional to $\mid T_N \mid^2$.

However, charged particles experience not only nuclear (strong) but also electromagnetic, weak and gravitational interactions. The last two are too weak and can be neglected in all realistic conditions while the electromagnetic one may become even stronger than the strong interaction if to consider low enough transferred momenta (Coulomb scattering)[1].

Thus, an essential interference takes place only between strong (N) and Coulomb (C) contributions. As we know well the Coulomb part, we can try to extract from the data an information on phase.

Normal way: first, one considers the region of transferred momenta where Coulomb contribution is negligible (due to fast decrease of form factors) and fixes $\mid T_N \mid$ by a suitable fitting procedure. Then this is used in the region of low transferred momenta where both strong and electromagnetic contributions are essential and where the strong interaction phase,

[1]In general electromagnetic interaction reveals itself (beyond Coulomb exchanges) in non controllable radiation of soft photons which makes impossible pure elastic scattering of charged particles. With account of virtual exchanges this effect gives an inclusive process where energies of undetected quanta are confined to the energy resolution of the colliding charged particles and is quantified in a well-known damping factor depending on the transferred momenta [2].In our case of very low transferred momenta this factor is practically equal to unity and we don't take it into account.

$Arg\, T_N(s,t)$, reveals due to the interference. As this Coulomb-nuclear interference(CNI)occurs in quite a narrow region of very low transferred momenta normally the information on the phase concerns $ArgT_N(s,0)$ only. When processing the data the quantity

$$\rho = \cot ArgT_N(s,0)$$

is usually presented. The quantity $\rho$ is related to the retrieval of the total cross-section (see the next Section for more detail.)

It is important to clearly realize that these "experimental" data on $\rho$ are only half those, since the need to preliminary know the amplitude modulus implicitly introduces dependence on the model that was used to process the data with $t$ outside the CNI region.

The retrieval of $\rho$ would be a routine operation if one would possess a consistently derived and universally recognized formula which combines both strong (N) and Coulomb(C) interactions in a single amplitude $T_{C+N}(s,t)$.

However, that is not the case. Till the recent times several formulas for $T_{C+N}(s,t)$ were in use "on the market". Most popular at the moment are formulas due to an early paper by H.Bethe [3](with several later modifications) and by R.Cahn [4](with later modifications due to Ref.[5]). Both formulas were contested in Ref.[6] where inconsistencies of papers [3],[4] and [5] were identified and a corresponding modification of formulas from Refs.[4],[5] was given. As was shown in Ref.[7], the use of the modified formula leads to values of $\rho$ different than those obtained with help of formulas from [3],[4],[5].

The motivation for the present work is that all mentioned above papers dealt with impact parameter representation which, in some respects, is not quite rigorous and thus a need in more rigorous derivation of the results of Ref.[6], if to take into account the conceptual importance of the topic, seems quite relevant and mandatory.

The approach used in this work is based on application of expansions in Legendre polynomials which have a rigorous mathematical ground.

Before proceeding to a straightforward presentation of our theoretical derivations and results, we found it useful to give a concise overview of the experimental results related to the topic of the present dissertation.

# Chapter 2

# Experimental results

As was said above, the $\rho$ parameter value can be obtained from the differential cross-sections by the virtue of the effects of Coulomb-nuclear interference (CNI). This parameter is important for more precise determination of the total cross-section. Here we limit ourselves by commenting only experiments made by the collaboration TOTEM at the LHC at the energy 13 TeV [8]. In Ref.[8] the following formula was taken for the cross-section due to the Coulomb scattering only

$$\frac{d\sigma_C}{dt} = \frac{1}{16\pi s^2} \mid T_C \mid^2 = \frac{4\pi\alpha^2}{t^2} F^4(t) \tag{2.1}$$

where $\alpha$ is the fine structure constant, $F(t)$ is the experimentally determined electromagnetic form-factor. In the paper [8], several choices for the form-factors have been examined and no difference has been noticed in the result. The modulus of nuclear amplitude $T_N(t)$ at low $|t|$ which was chosen in [8] is given by the following expression

$$|T_N(t)| = \sqrt{\frac{s}{\pi}} \frac{p}{\hbar c} \sqrt{a} exp(\sum_{n=1}^{N_b} b_n t^n) \tag{2.2}$$

where

$$a = d\sigma_N/dt \mid_{t=0} .$$

The parameter $b_1$ is responsible for the leading exponential decrease, the other $b_n$ parameters were added to describe small deviations from the pure exponential. Extending the nuclear amplitude to higher values of $|t|$ is worth because the CNI formula (Eq.(2.4) below) involves integrations. The nuclear amplitude at larger $|t|$ was modelled by a function that gives the dip-bump structure observed in the data in the region $\mid t \mid \approx 0.5 GeV^2$ (Fig.2.1).

The intermediate $|t|$ part was provided with continuous and smooth inter-

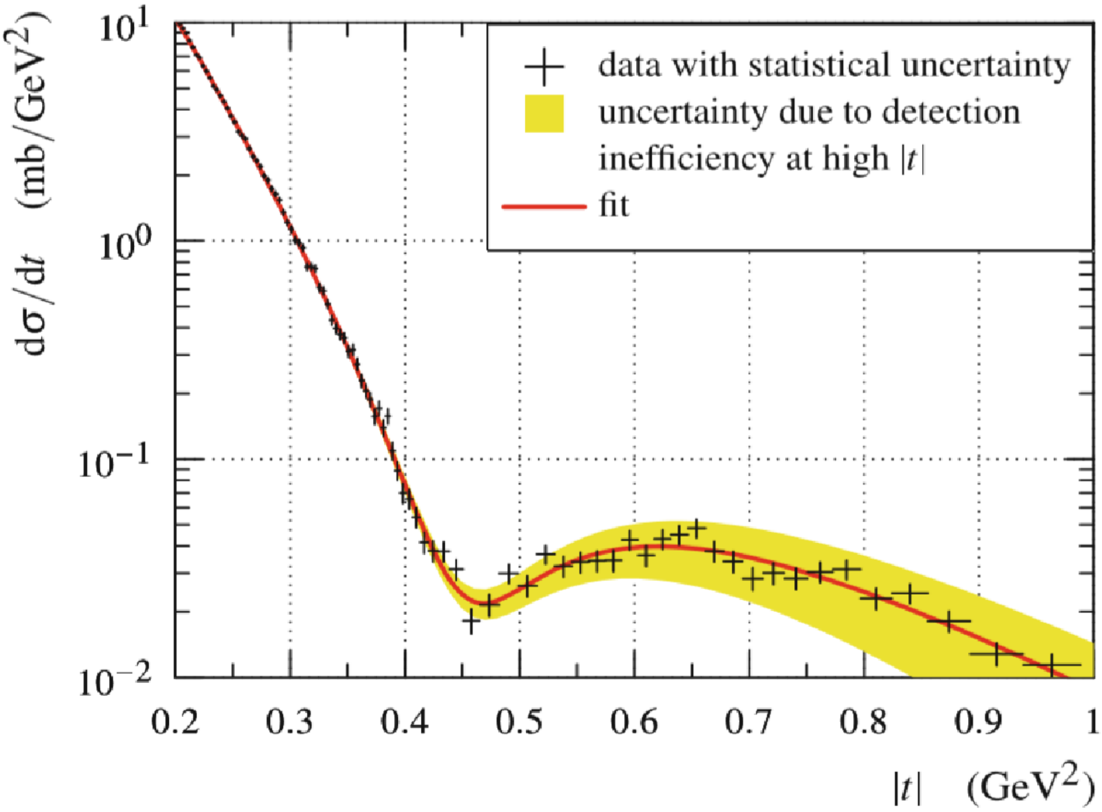


Figure 2.1: Differential cross-section of elastic pp scattering beyond the CNI region(for details see Ref.[8]).

polations between low and high $|t|$ regions.

It has been noticed in [8], however, that "changing high $|t|$ part within reasonable limits has almost no impact on the results".

The differential cross-section zoomed in the region of low $t$, where CNI is essential,is depicted at Fig.2.2

It was mentioned in [8] that several parametrisations have been considered for the phase of the amplitude of nuclear interactions, both $t$ dependent and constant. However, "no dependence on this choice was observed and therefore only the constant phase

$$ArgT_N(t) = \frac{\pi}{2} - arctan\rho = const \tag{2.3}$$

was retained".

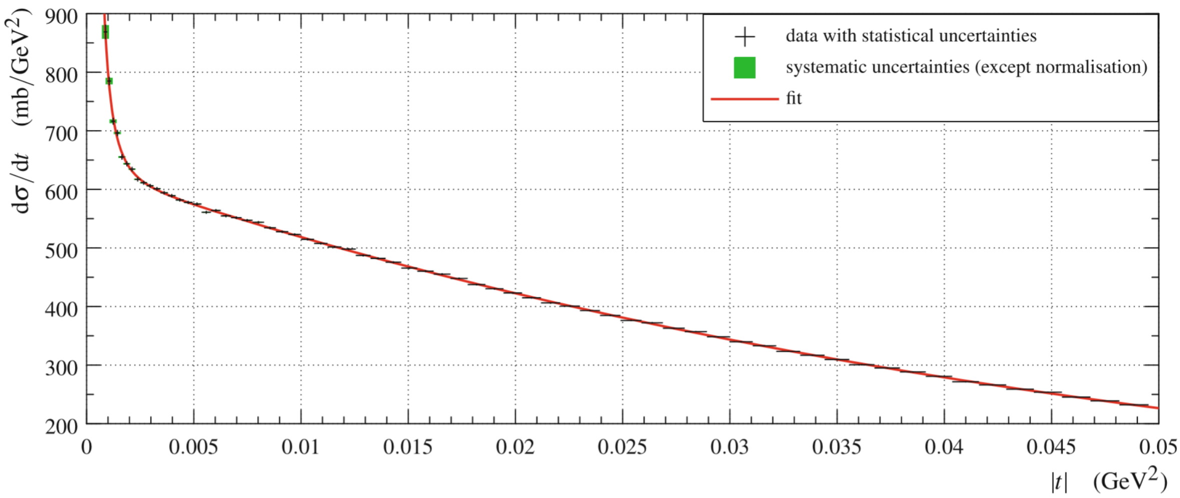


Figure 2.2: TOTEM fit to the differential cross-section in the region of CNI.

In paper[8] there was used the interference formula given in [5]:

$$\begin{aligned} \frac{d\sigma_{C+N}}{dt} &= \frac{\pi(\hbar c)^2}{sp^2}|\frac{\alpha s}{t}F^2 + T_N[1 - i\alpha G(t)]|^2, \\ G(t) &= \int_{-4p^2}^{0} dt' log\frac{t'}{t}\frac{d}{dt'}F^2(t') \\ &- \int_{-4p^2}^{0} dt'(\frac{T_N(t')}{T_N(t)} - 1)\frac{I(t,t')}{2\pi}, \\ I(t,t') &= \int_0^{2\pi} d\phi\frac{F^2(t'')}{t''}, \\ t'' &= t + t' + 2\sqrt{tt'}cos\phi. \end{aligned} \tag{2.4}$$

which is quite similar to that given in Ref.[4].

For retrieving the total cross-section there was used the formula following from the optical theorem:

$$\sigma_{tot}^2 = \frac{16\pi(\hbar c)^2}{1 + \rho^2}a \tag{2.5}$$

The use of Eqs.(2.4) and (2.5) for analysis of the TOTEM experimental data on $d\sigma_{C+N}/dt$ resulted in the following values of parameters $\rho$ and $\sigma_{tot}$ in proton-proton collisions at 13 TeV:

$$\rho = 0.10 \pm 0.01 \quad \sigma_{tot} = 110.5 \pm 2.4mb. \tag{2.6}$$

Such values, mostly that of $\rho$, were interpreted in the TOTEM publication [8] as

1.*the evidence of "a C-odd three-gluon compound state"*

and

2. *the indication of a slow down of the growth rate of $\sigma_{tot}$* .

General view of the total, elastic and inelastic cross-section is presented at Fig.2.3

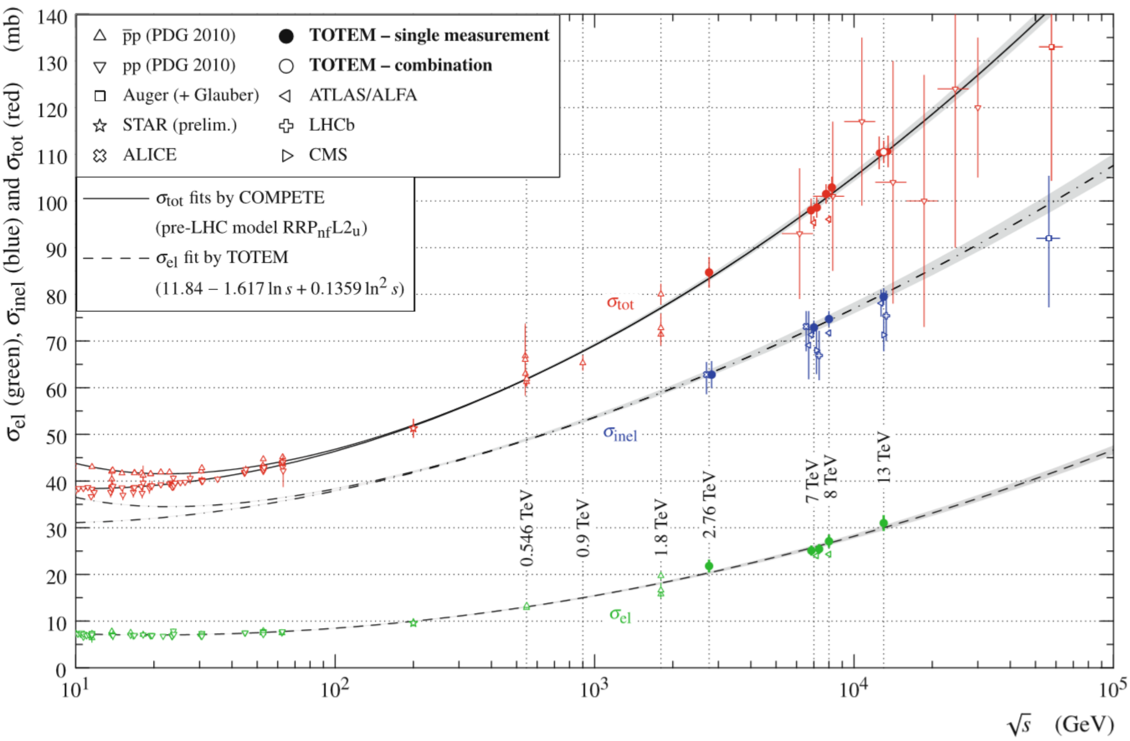


Figure 2.3: Energy evolution of the total, elastic and inelastic cross-section at $\sqrt{s} = 10 \div 13000$ GeV. The solid line is due to the COMPETE parametrization[9].

Energy evolution of the $\rho$ is presented at Fig.2.4

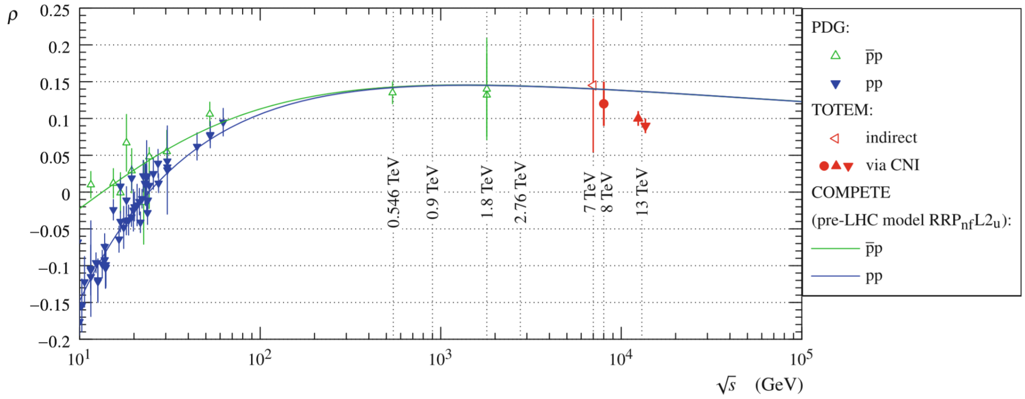


Figure 2.4: Energy evolution of the $\rho$ parameter at $\sqrt{s} = 10 \div 13000$ GeV. The solid line is the prediction of the COMPETE parametrization [9].

These results and conclusions from them made in [8] initiated a vivid (often quite critical) discussion resulted in several tens of publications.

# Chapter 3

# Proton-proton scattering amplitude with account of Coulomb exchanges. General Formula.

Hereinafter, as the scattering variable, we use the cosine of the scattering angle in the center of mass system.

$$z = cos\theta = 1 + \frac{t}{2p^2}$$

The strong interaction elastic scattering amplitude $T_N(s, z)$ can be expanded into the series in Legendre polynomials

$$T_N(s, z) = \sum_{l=0}^{\infty} (2l + 1) P_l(z) T_N^l(s)$$

which converges homogeneously and uniformly inside the elliptical region [10]

$$| 1 - z | + | 1 + z | \leq 2x_0(s)$$

where the value of the major semi axis $x_0$ depends on the masses of colliding particles and generically behaves at high energies as $x_0 \approx 1 + const/s$.

In our normalization, the Holtzmark expansion in Legendre polynomials taking into account the symmetry of the scattering for $\theta \to \pi - \theta$ have the form (Our normalization here differs from that used in Chapter 2 by factor 2.):

$$T_X(s, z) = \frac{1}{2} \sum_{l=0}^{\infty} (2l + 1) \frac{1 + (-1)^l}{2} P_l(z) \frac{1}{2i} (e^{2i\delta_X^l(s)} - 1) \tag{3.1}$$

where $X = (C+N), C, N$ and $\delta_X$ is the partial wave phase shift due to exchange X. In this normalization the differential cross-section takes the form

$$\frac{d\sigma_{C+N}}{dt} = 4\pi\frac{|T_{C+N}|^2}{sp^2}$$

Since the differential cross-section is defined only by modulus of amplitude $T_{C+N}$, we will continue to deal with it. The relative phases of the Coulomb and strong amplitude of course affect its value.

Taking into account that all the results for experimental research on " forward/backward scattering" i.e. for $\theta = 0$ or $\pi$, are in fact the results of extrapolation $z = \cos\theta \to \pm 1$, and using the identity

$$\sum_{l=0}^{\infty}(2l+1)P_l(z)P_l(z') = 2\delta(z-z')$$

we obtain from formula (3.1) the following fundamental relation,

$$|T_{C+N}|_{|z|\neq 1} = |\frac{E_C(z)}{2i} + \frac{1}{4}\int_{-1}^{1} dz'dz''K(z,z',z'')E_C(z')T_N(z'')|, \tag{3.2}$$

where,

$$E_C(z) = \frac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(z)\frac{1+(-1)^l}{2}e^{2i\delta_C^l}$$

$$K_+(x,y,z) = \sum_{l=0}^{\infty}(2l+1)P_l(x)P_l(y)P_l(z) = \frac{2}{\pi}(1+2xyz+x^2-y^2-z^2)^{(-1/2)}$$

$$T_N(z) = \sum_{l=0}^{\infty}(2l+1)P_l(z)\frac{1+(-1)^l}{2}\frac{(e^{2i\delta_N^l}-1)}{2}$$

here,

$$\delta_C^l = \int_{-1}^{1} dz T_C^{Born}(s,z)P_l(z), \tag{3.3}$$

where,

$$T_C^{Born}(s,|z|\to 1) \approx -\frac{\alpha}{2}\frac{F^2(-2p^2(1-|z|))}{1-|z|+\lambda^2/2p^2} \tag{3.4}$$

$F(t(u))$ is the electromagnetic form factor of proton, and $\lambda$ is a fictitious mass of photon as a regulator of infrared divergence.
As we can see from (3.2) by condition $|z| \neq 1$, singular dependence on $\lambda$ gets factorized.

If we represent $\delta_C^l$ in the form

$$\delta_C^l = \int_{-1}^{1} dz T_C^{Born}(s,z)[P_l(z)-1] + \int_{-1}^{1} dz T_C^{Born}(s,z) = \hat{\delta}_C^l + \delta_\lambda$$

where,

$$\delta_\lambda = \int_{-1}^{1} dz T_C^{Born}(s,z)$$

then,

$$|T_{C+N}|_{|z|\neq 1} = |(e^{2i\delta_\lambda})\hat{T}_{C+N}| = |\frac{\hat{E}_C(z)}{2i} + \frac{1}{2}\int_{-1}^{1} dz' dz'' K(z,z',z'')\hat{E}_C(z')T_N(z'')|, \tag{3.5}$$

here,

$$\hat{E}_C(z) = \frac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(z)\frac{1+(-1)^l}{2}e^{2i\hat{\delta}_C^l}$$

In equation (3.5) we may take the physical photon mass value $\lambda = 0$, so that the term $|T_{C+N}|_{|z|\neq 1}$ does not contain non-physical parameters.

Thus, formula (3.5) is an exact (all orders in $\alpha$) expression for the modulus of the physical scattering amplitude of identical charged particles with account of Coulomb - nuclear interference. After having chosen a particular model for $T_N$ and $F(t)$ equation (3.5) can be used for estimating the value of the differential cross-section. This mainly concerns the region of transferred momenta of the order $10^{-3}$ - $10^{-4}$ $GeV^2$, where Coulomb - nuclear interference becomes noticeable.

However, in practice one normally uses one or another approximation in $\alpha$. This will be the subject of our consideration in Chapters 4 and 5.

# Chapter 4

# Coulomb - nuclear interference in the lowest order in $\alpha$.

Quite often first-order approximations in $\alpha$ are used ([4],[5]). It was shown in Ref.[6] that expansion of Eq.(3.5) leads to the expression which does not contain an extra term which was obtained in corresponding approximation considered in Ref.[5]. This result was obtained with use of the impact parameter representation which holds only approximately (for very high energies). In this Chapter we will check the result obtained in Ref.[6] with help of rigorously proved expansions in series in Legendre polynomials.
Function $\hat{E}_C(z)$ has the following form when expanded in the series in $\alpha$:

$$\hat{E}_C(z) = \delta(z^2-1)+2i\int_{-1}^{1} dz' T_C^{Born}(z')(|z'|\delta(z'^2-z^2)-\delta(z^2-1))+O(\alpha^2) \quad (4.1)$$

for $|z| \neq 1$ this expression takes the form:

$$\hat{E}_C(z) = 2iT_C^{Born}(z)$$

general result:

$$|T_{C+N}|_{|z|\neq 1} = |T_C^{Born}(z) + T_N(s,z) - i\int_{-1}^{1} dz' J_C(z,z')[T_N(s,z) - T_N(s,z')]|, \quad (4.2)$$

where,

$$J_C(z,z') = \int_{-1}^{1} d\zeta K(z,z',\zeta) T_C^{Born}(\zeta) \quad (4.3)$$

This approximation is more convenient for phenomenological purposes.
For point-like charges ( i.e. F = 1 )

$$J_C(z,z') = -\frac{\alpha}{2}(\frac{1}{|z-z'|} + \frac{1}{|z+z'|})$$

In (4.2) the singularity of general function $\frac{1}{z \pm z'}$ is eliminated by the actual regularization with the help of amplitude $T_N(s,z)$, performing the role of test function of z, so far $T_N(s,z) \in C^\infty([-1-\epsilon, 1+\epsilon])$ for $|z| \le 1$.
In applications, the use of the scattering angle has long since given way to invariant variables t(u). Therefore, we present our result (4.2) in terms of invariant variables.
The most relevant invariant variable accounting for the forward-backward symmetry of the amplitude is the square of the 2D vector $\mathbf{q} = (q_1, q_2)$ transverse to the beam direction

$$q^2 = q_1^2 + q_2^2 = ut/4$$

In these variables Eq.(4.2) acquires the form ( we omit the explicit indication of the energy argument, s):

$$|T_{C+N}|_{|q|\neq 0} = |T_C^{Born}(q^2) + T_N(s,q^2) + i\alpha \int_0^\infty dq'^2 \frac{F^2(q'^2)}{q'^2}[T_N(q^2) - \bar{T}_N(q^2, q'^2)]| \tag{4.4}$$

where,

$$T_C^{Born}(q^2) = -\frac{\alpha s F(q^2)}{2q^2}$$

and

$$\bar{T}_N(q^2, q'^2) = \int_0^{2\pi} \frac{d\phi}{2\pi} T_N(q^2 + q'^2 - 2qq' cos\phi)$$

Thus, our formula (4.4) verifies the result of the first order from [5] in the positive.

# Chapter 5

# Coulomb - nuclear interference: the importance of second order terms.

As already mentioned above, the 1st order approximations for $T_{C+N}$ were used for extraction of the parameter $\rho(s) = ReT_N(s,t=0)/ImT_N(s,t=0)$ from the data on $d\sigma/dt \sim |T_{C+N}|^2$ [8].
The 1st order approximation obtained in Ref.[6] essentially differs from that from Ref.[5] (note that in Ref.[8] the results of Ref.[5] was used).
It is, however, clear from the point of view of simple mathematical consistency that if one considers the 1st order approximation for $T_{C+N}$ then the same 1st order approximation should be used for its modulus, $|T_{C+N}|$ as well.
It is evident also that if we take two amplitudes $T'_{C+N}$ and $T''_{C+N}$ which differ from each other by a phase factor only, i.e.

$$T''_{C+N} = e^{i\alpha L} T'_{C+N} \tag{5.1}$$

where L is an arbitrary real function of s and $cos\theta$ (in physical regions of both variables), then

$$|T''_{C+N}| = |T'_{C+N}|.$$

Thus, both amplitudes are physically equivalent and anyone of the two can be used. Let us now take the 1st order approximations:

$$T'_{C+N} = T_N + \alpha A'$$

$$T''_{C+N} = T_N + \alpha A''$$

It follows from (5.1) that

$$A'' = A' + iLT_N \tag{5.2}$$

Thus, if two amplitudes $T'_{C+N}$ and $T''_{C+N}$ have their 1st order approximations which are different but are related by relationship (5.2) then these amplitudes are equivalent, i.e. their moduli coincide. We have, nonetheless, to keep in mind that this is guaranteed in the 1st order approximation only.
Let us notice that the difference in the 1st-order approximations to $T_{C+N}$ presented in Ref. [6] and [5] is exactly of the type (5.2).
However, the values of the parameter $\rho(s)$ extracted from the data with help of these amplitudes using the same nuclear amplitude $T_N$ appeared essentially different [7]. What is the reason for such a discrepancy?
The reason is that these amplitudes were used beyond the limits of validity of their approximations. In fact, as was mentioned above, the observed cross-sections are proportional not to $|T_{C+N}|$ but to $|T_{C+N}|^2$. When extracting the parameter $\rho$ the terms authors of Ref.[6] and [8] obtained $|T_{C+N}|^2$ squaring only the 1st order approximation for $|T_{C+N}|$ without accounting for terms $\sim \alpha^2$. Such an action is certainly mathematically inconsistent. To be consistent, one must retain terms $\sim \alpha^2$ in the expansion of the amplitude itself (or of its modulus).
Let us illustrate the said. Let us use the 1st-order expansion

$$T_{C+N} = T_N + \alpha A$$

for obtaining $|T_{C+N}|^2$. We get

$$|T_{C+N}|^2 = |T_N|^2 + \alpha 2Re(T_N^* A) + \alpha^2 |A|^2.$$

Let us now take a more exact expansion of $T_{C+N}$:

$$T_{C+N} = T_N + \alpha A + \alpha^2 B$$

We get now retaining – as it should be – the terms not higher than $\sim \alpha^2$

$$|T_{C+N}|^2 = |T_N|^2 + \alpha 2Re(T_N^* A) + \alpha^2(|A|^2 + 2Re(T_N^* B))$$

We see that uncritical use of the 1st order expressions leads to missing terms $\alpha^2 2Re(T_N^* B)$ which can be, dependent on the considered region of the kinematical variables, quite compatible with old terms $\alpha^2(|A^2|)$. Note that this term contains the contribution from the pure Coulomb scattering which dominates only at extremely low values of transferred momenta of order $\leq 10^{-4}$ $GeV^2$.
In which case two amplitudes $T'_{C+N}$ and $T''_{C+N}$ will be equivalent at the level of the 2nd order accuracy?
The following two conditions should now hold:

$$A'' = A' + iLT_N \tag{5.3}$$

$$B'' = B' + iLA' - \frac{1}{2}L^2 T_N \tag{5.4}$$

One can show [11] that while $\alpha$ expansion coefficients of the amplitudes used in Refs.[6] and [5] obey to Eq.(5.3) they do not obey to Eq.(5.4).

Now we demonstrate the explicit expression of the second-order coefficients in the expansion

$$T_{C+N} = T_N + \alpha A + \alpha^2 B$$

The first order coefficient was already presented and discussed in Chapter 4. So we concentrate on the coefficient B.
It is much more complicated that the coefficient A which is of a relatively simple form:

$$\alpha A = T_C^{Born}(z) - i\int_{-1}^{1} dz' J_C(z,z')[T_N(s,z) - T_N(s,z')]$$

After quite cumbersome transformations, the second order term can be cast in the form

$$\begin{aligned}\alpha^2 B =& \frac{i}{2}\int_0^1 dz''dz' K(z',z'',z) T_C^{Born}(z') T_C^{Born}(z'') - T_C^{Born}(z)[T_C^{Born}(z') + T_C^{Born}(z'')] \\ &- \int_{-1}^{1} dz'[M(z,z')T_N(s,z') - N(z')T_N(s,z')]\end{aligned} \tag{5.5}$$

Here

$$M(z,z') = \int_{-1}^{1} dz_1 dz_2 T_C^{Born}(z_1) T_C^{Born}(z_2)[K(z,z',z_1) + K(z,z',z_2) - I(z,z',z_1,z_2)]$$

$$N(z') = \int_{-1}^{1} dz' dz'' T_C^{Born}(z_1) T_C^{Born}(z_2) K(z',z_1,z_2);$$

$$I(z,z',z_1,z_2) = \sum_{l=0}^{\infty}(2l+1)P_l(z)P_l(z')P_l(z_1)P_l(z_2)$$

From definition of the function $K$ we readily obtain the following relation

$$I(z,z',z_1,z_2) = \frac{1}{2}\int_{-1}^{+1} d\zeta K(z,z',\zeta)K(\zeta,z_1,z_2)$$

If we go to the commonly used variables $t = -2p^2(1-z)$, then we get expressions that match (up to corrections $\sim t/p^2$ negligible in the CNI region) the expressions obtained in the framework of the representation of impact parameters in Ref.[11]. We do not write them here explicitly because of their bulkiness.

# Chapter 6

# Conclusions

This work is devoted to problems related to the account of the Coulomb-nuclear interference in hadron scattering.
We have formulated conditions at which different amplitudes are physically equivalent in correspondence with the degree of approximation.
We have calculated for the first time the terms of the second order which are important as discussed above in selecting correct amplitudes. This result holds at any energy.
Expressions (4.2) and (5.5) are the main result of this work. Their advantage in comparison with similar formulas obtained in Refs [6], [11] with use of approximate impact parameter representation is that they are obtained on basis of exact and rigorously proved expansions in Legendre polynomials. We thus have verified the validity of formulas obtained in Refs [6], [11] at ultra relativistic energies when $\mid t \mid /s \ll 1$.

So, they can be used for extraction of the parameter $\rho$ from the high-energy data (e.g. LHC) which plays an important role in discriminating various models of strong interactions.

# Chapter 7

# Appendix Expansion of scattering amplitude in Legendre polynomials and its properties [12]

## 7.1 Introduction and properties

In this section we give some basic introduction to Legendre polynomials, as the expansion of the amplitude in above paper is given in terms of these polynomials. These polynomials can occur as a solution of the Legendre ODE, as a consequence of the Rodrigues formula or originate as the complete set of orthogonal functions in the $[-1, 1]$ (*Gram Schmidt orthogonalization*). These polynomials are obtained by their corresponding generating functions.

## 7.2 Physical Basis — Electrostatics

We here use a physical example to define a generating functions and then get then provide a method to obtain the Legendre function from it. Consider a unit electric charge placed on the z-axis at z=a. As shown in Fig.(7.1), the electrostatic potential of unit charge is

$$\phi = 1/r_1 \quad (In\ Gaussian\ units) \tag{7.1}$$

Writing the potential in terms of spherical coordinates r and $\theta$.
Using the cosine law in Fig.(7.1), we get

$$\phi = (r^2 + a^2 - 2arcos\theta)^{-1/2} \tag{7.2}$$

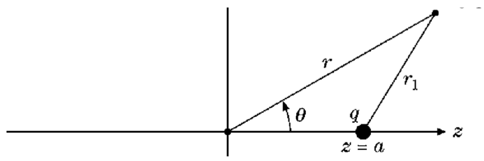


Figure 7.1: Electrostatic potential. Unit charge displaced from origin.

## 7.3 Legendre Polynomials

For the case of $r > a$ or, more precisely, $r^2 >| a^2 - 2ar\cos\theta |$. The eq.(7.2) can be expanded in the powers of $(\frac{a}{r})$:

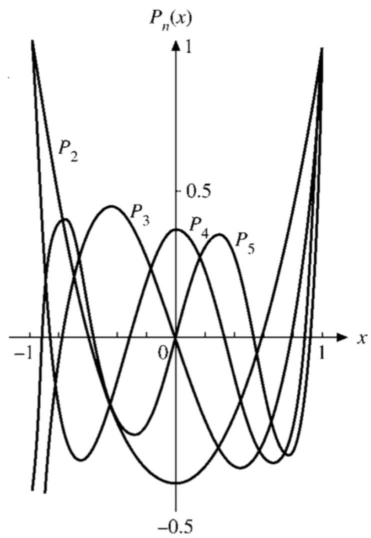


Figure 7.2: Legendre polynomials $P_2(x), P_3(x), P_4(x), and P_5(x)$.

$$\phi = \sum_{n=0}^{\infty} P_n(\cos\theta)(\frac{a}{r})^n \tag{7.3}$$

Replacing $x$ and $t$ with $\cos\theta$ and $a/r$ , respectively, we have

$$g(t,x) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{7.4}$$

Eq.(7.4) is our generating function formula. Using binomial expansion it can be shown that $| P_n(\cos\theta) |\leq 1$, which means that the series expansion

Eq.(7.4) is convergent for $| t |< 1$. Indeed, the series is convergent for $| t |= 1$ except for $| x |= 1$.

## 7.4 Orthogonality

Legendre's differential equation can be written in the form:

$$\frac{d}{dx}[(1-x^2)P'_n(x)] + n(n+1)P_n(x) = 0 \tag{7.5}$$

showing clearly that it is self-adjoint. If it satisfies certain boundary conditions, then we know that the solutions of the above equation $P_n(x)$ will be orthogonal. For m $\neq$ n

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, m \neq n \tag{7.6}$$

$$\int_{0}^{\pi} P_n(cos\theta)P_m(cos\theta)sin\theta d\theta = 0, m \neq n \tag{7.7}$$

which shows that $P_n(x)$ and $P_m(x)$ are orthogonal in the interval $[-1, 1]$. We shall need to evaluate the integral ($Eq.$ (7.6)) when $n = m$. Certainly it is no longer zero. From our generating function,

$$(1-2xt+t^2)^{-1} = [\sum_{n=0}^{\infty} P_n(x)t^n]^2 \tag{7.8}$$

Integrating from x =-1 to x =+1, we have

$$\int_{-1}^{1} \frac{dx}{1-2tx+t^2} = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} [P_n(x)]^2 dx \tag{7.9}$$

the cross terms in the series vanish by means of $Eq.$ (7.7). Using $y = 1 - 2xt + t^2$, $dy = -2tdx$, we obtain

$$\int_{-1}^{1} \frac{dx}{1-2tx+t^2} = \frac{1}{2}\int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = \frac{1}{t} ln\frac{1+t}{1-t} \tag{7.10}$$

Expanding this in a power series gives us

$$\frac{1}{t} ln\frac{1+t}{1-t} = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} \tag{7.11}$$

Comparing power-series coefficients of $(Eq.\,(7.9)\ \ and\ Eq.\,(7.11))$ , we must have

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{2}{2n+1} \tag{7.12}$$

Combining $(Eq.\,(7.7)\ \ with\ Eq.\,(7.12))$ we have the orthonormality condition

$$\int_{-1}^{1}[P_m(x)][P_n(x)]dx = \frac{2\delta_{mn}}{2n+1} \tag{7.13}$$

## 7.5 Expansion of Functions, Legendre Series

Legendre polynomials are orthogonal and forms a complete set Let the series,

$$\sum_{n=0}^{\infty} a_n P_n(x) = f(x) \tag{7.14}$$

converges in the mean in the interval $[-1, 1]$. The coefficients $a_n$ can be found by multiplying the series by $P_m(x)$ and integrating term by term. Using the orthogonality property expressed in $(Eq.\,(7.7)\ \ and\ Eq.\,(7.13))$, we obtain

$$\frac{2}{2m+1}a_m = \int_{-1}^{1} f(x)P_m(x)dx \tag{7.15}$$

We replace the variable of integration x by t and the index m by n. Then, substituting into $(Eq.\,(7.14))$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2}(\int_{-1}^{1} f(t)P_n(t)dt)P_n(x) \tag{7.16}$$

This expansion in a series of Legendre polynomials is usually referred to as a Legendre series.

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